

A CONTRIBUTION TO THE FONG-TSUI CONJECTURE RELATED TO SELF-ADJOINT OPERATORS

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ABSTRACT. We are interested in an open question raised by Fong-Tsui (dating back to the beginning of the eighties of last century) as to whether a bounded operator whose absolute value is less than the absolute value of its real part is self-adjoint. The analogue in the unbounded operators setting is also treated.

1. NOTATIONS AND TERMINOLOGY

If T is a linear operator on a complex Hilbert space, then $\operatorname{Re}T$ denotes the real part of T , that is, the operator $\frac{T+T^*}{2}$, where T^* is the adjoint of T .

The absolute value of T , denoted by $|T|$, is the positive square root of the positive operator T^*T . Normal, self-adjoint, positive operators and isometries are defined in their usual fashion.

If T is an operator such that $T + T^*$ commutes with T^*T , then we say that T belongs to the Θ -class (a class of operators introduced by S.L. Campbell, see [1]).

For basic results on operator theory, the reader may consult [2] or [5].

2. ESSENTIAL BACKGROUND

In the present section we recall the main results which will be needed to prove our theorems.

Theorem 1 ([3]). *If T is a bounded operator verifying $|T|^2 \leq (\operatorname{Re}T)^2$, then T is automatically self-adjoint.*

Theorem 2 ([4]). *If T is a bounded operator satisfying $|T| \leq \operatorname{Re}T$, then T is positive.*

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3. POSITIVE RESULTS

The following conjecture appeared in [4] (which was a try to strengthen a result which appeared in [3])

Conjeture (Fong-Tsui [4]). *If T is a bounded operator and $|T| \leq |\operatorname{Re}T|$, then T is self-adjoint.*

The authors in [4] gave some cases where this result holds, eg if T is compact or if the Hilbert space is finite-dimensional (among others).

In the present paper we provide a modest contribution to the conjecture in order to maintain the hope towards a complete solution of it. We first give some classes for which the conjecture is true, then, and in the end, we give an unbounded counterexample that shows that the conjecture does not hold for unbounded operators.

Here is the first result:

Theorem 3. *Let T be a linear operator on a Hilbert space. Let U be the unitary operator intervening in the polar decomposition of the self-adjoint operator $\operatorname{Re}T$. If $UT = TU^*$, then $|T| \leq |\operatorname{Re}T|$ implies that T is self-adjoint.*

Proof. Let U be a unitary operator such that $\operatorname{Re}T = U|\operatorname{Re}T|$. We may then write

$$|T| \leq |\operatorname{Re}T| = U^*\operatorname{Re}T.$$

From the hypothesis $UT = TU^*$ and by the unitarity of U (or by the Fuglede theorem), we get $U^*T = TU$. Moreover, we obtain

$$U^*\operatorname{Re}T = \operatorname{Re}(U^*T).$$

Observe that

$$|U^*T| = \sqrt{(U^*T)^*(U^*T)} = \sqrt{T^*UU^*T} = \sqrt{T^*T} = |T|.$$

So the hypothesis $|T| \leq |\operatorname{Re}T|$ now looks like

$$|U^*T| \leq \operatorname{Re}(U^*T).$$

Theorem 2 yields the positiveness of U^*T or simply its self-adjointness. Calling on again the hypothesis $UT = TU^*$, we immediately see that

$$T^*U = (U^*T)^* = T^*U = U^*T = TU,$$

which implies that T is self-adjoint thanks to the unitarity of U . This finishes the proof. \square

The condition $UT = TU^*$ in the previous theorem can be dropped at the cost of assuming that T is in the Θ -class. We have:

Theorem 4. *Let T be a bounded operator belonging to the Θ -class. Then T is self-adjoint whenever $|T| \leq |\operatorname{Re}T|$.*

Proof. Since $T+T^*$ and T^*T commute, $\operatorname{Re}T$ commutes with $|T|$. Hence the self-adjointness of $\operatorname{Re}T$ implies that $|T|$ commutes with $|\operatorname{Re}T|$, i.e.

$$|T||\operatorname{Re}T| = |\operatorname{Re}T||T|.$$

Since $|T| \leq |\operatorname{Re}T|$ (and $|T| \geq 0$!), the previous displayed equation implies that

$$|T|^2 \leq |T||\operatorname{Re}T| \text{ and } |T||\operatorname{Re}T| \leq |\operatorname{Re}T|^2 = (\operatorname{Re}T)^2$$

or just

$$|T|^2 \leq (\operatorname{Re}T)^2.$$

The self-adjointness of T now follows from Theorem 1, completing the proof. \square

It is obvious that normal operators T are in the Θ -class. Therefore we have the

Corollary 1. *Let T be a normal operator such that $|T| \leq |\operatorname{Re}T|$. Then T is self-adjoint.*

Similarly, isometries do lie in the Θ -class and so we have the

Corollary 2. *Let T be an isometry operator verifying $|T| \leq |\operatorname{Re}T|$. Then T is self-adjoint.*

4. THE UNBOUNDED ANALOGUE

In this section we present an *unbounded* counterexample to the conjecture. This was probably expected due to the hazardous terrain of domains of unbounded operators. Here is the counterexample:

Let S be an unbounded operator S , defined in a Hilbert space \mathcal{H} say, with domain $D(S) \subsetneq \mathcal{H}$. Set $T = S - S$. Then $T = 0$ on $D(T) = D(S) \subsetneq \mathcal{H}$.

It is clear that T *cannot be self-adjoint* as it is not closed (it is, however, symmetric) and that $D(T^*) = \mathcal{H}$. Nonetheless, we have the following

$$T^*T(f) = 0 \text{ on } D(T^*T) = \{f \in D(T) : 0 \in D(T^*)\} = D(T)$$

and

$$\left(\frac{T+T^*}{2}\right)(f) = 0 \text{ on } D(T) \cap D(T^*) = D(T) \text{ as } D(T^*) = \mathcal{H}$$

which lead to *formally* write $|T| = |\operatorname{Re}T|$.

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